Variation of Parameters –

This method, in theory, can be used to find a particular solution of $y''+p(x)y'+g(x)y=f(x)$ for any $f(x)$. (Recall undetermined coefficients requires specific forms). In practice you may not be able to perform the necessary integrations.

Suppose we have the linear 2^{nd} order equation –

$$y''+p(x)y'+g(x)y=f(x)$$

and we know

$y_g=c_1y_1+c_2y_2$ is the general solution of the homogeneous equation

$$y''+p(x)y'+g(x)y=0$$

(Remember! This means that y_1 and y_2 are each solutions!)

Similar to what we've done before, we assume

$$y_p=y_1(x)v_1(x)+y_2(x)v_2(x) \quad \text{(we can try anything!)}$$

We will also need a second condition, that is

$$(*) \quad v_1'(x)y_1(x)+v_2'(x)y_2(x)=0$$

then $y_p'=y_1'v_1+y_1v_1'+y_2'v_2+y_2v_2'$

$=y_1'v_1+y_2'v_2$ after applying condition $(*)$

$y_p''=y_1''v_1+y_1'v_1'+y_2''v_2+y_2'v_2'$

If y_p satisfies our equation then –

$$y_1''v_1+y_1'v_1'+y_2''v_2+y_2'v_2'+p(x)\{y_1'v_1+y_2'v_2\}+g(x)\{y_1v_1+y_2v_2\}=f(x)$$

$$v_1\{y_1''+p(x)y_1'+g(x)y_1\}+v_2\{y_2''+p(x)y_2'+g(x)y_2\}+y_1'v_1'+y_2'v_2'=f(x)$$

or $\quad y_1'v_1'+y_2'v_2'=f(x)$ because y_1 and y_2 both satisfy $y''+p(x)y'+g(x)y=0$

Then combining this result with condition $(*)$ we have a system of two equations in v_1' and v_2', namely

$$y_1'v_1'+y_2'v_2'=f(x)$$
$$y_1v_1'+y_2v_2'=0$$

Which we can solve by substitution

$$v_1' = \frac{-y_2 v_2'}{y_1}$$

$$y_1'\left(\frac{-y_2 v_2'}{y_1}\right) + y_2' v_2' = f(x)$$

and $v_2' = \dfrac{y_1 f(x)}{y_1 y_2' - y_2 y_1'}$

$= \dfrac{y_1 f(x)}{W(y_1, y_2)}$ where $W(y_1, y_2)$ = the Wronskian of y_1 and y_2

then $v_1' = \dfrac{-y_2 v_2'}{y_1}$

$= \dfrac{-y_2 f(x)}{W(y_1, y_2)}$

and, by integration —

$$v_1 = -\int \frac{y_2 f(x)}{W(y_1, y_2)}\,dx \quad \text{and} \quad v_2 = \int \frac{y_1 f(x)}{W(y_1, y_2)}\,dx$$

If we can perform these integrations we have

$$y_p = y_1 v_1 + y_2 v_2$$

and $y = y_g + y_p$ is the general solution of the NH equation.

Ex: Note this is for illustration only. If undetermined coefficients is applicable you should choose that method!

$y'' - y' - 2y = e^{3x}$

$k^2 - k - 2 = 0$

$(k-2)(k+1) = 0$

and $y_g = c_1 e^{-x} + c_2 e^{2x}$ note here that $y_1 = e^{-x}$, $y_2 = e^{2x}$

$W(y_1, y_2) = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = 2e^{x} + e^{x} = 3e^{x}$

so $v_1 = -\int \dfrac{y_2 f(x)}{W(y_1, y_2)}\,dx = -\int \dfrac{e^{2x} \cdot e^{3x}}{3e^{x}}\,dx$

$= -\int \dfrac{e^{4x}}{3}\,dx = -\dfrac{e^{4x}}{12}$

and $v_2 = \int \dfrac{y_1 f(x)}{W(y_1, y_2)}\,dx = \int \dfrac{e^{-x} \cdot e^{3x}}{3e^{x}}\,dx$

$= \int \dfrac{e^{x}}{3}\,dx = \dfrac{e^{x}}{3}$

and $y = y_g + y_1 v_1 + y_2 v_2 = c_1 e^{-x} + c_2 e^{2x} - \frac{1}{12} e^{3x} + \frac{1}{3} e^{3x}$

$= \boxed{c_1 e^{-x} + c_2 e^{2x} + \frac{1}{4} e^{3x}}$ which was much easier by the previous method

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Ex: The utility of this method is when undetermined coefficients does not work.

$$y'' + y = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$y'' + y = 0$$

$$k^2 + 1 = 0$$

$$k = \pm i$$

So $y_g = c_1 \cos x + c_2 \sin x$ is the general solution of the homogeneous equation

$y_1 = \cos x$, $y_2 = \sin x$

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

Then if $y_p = y_1 v_1 + y_2 v_2$

$$v_1 = -\int \frac{y_2 \tan x}{W(y_1, y_2)} dx = -\int \sin x \tan x \, dx$$

$$= -\int \frac{\sin^2 x}{\cos x} dx$$

$$= \int \frac{\cos^2 x - 1}{\cos x} dx$$

$$= \sin x - \ln|\sec x + \tan x|$$

$$= \sin x - \ln(\sec x + \tan x) \quad \text{because } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$v_2 = \int \frac{y_1 \tan x}{W(y_1, y_2)} dx = \int \cos x \tan x \, dx$$

$$= -\cos x$$

So $y_p = y_1 v_1 + y_2 v_2$

$$= \cos x \sin x - \cos x \ln(\sec x + \tan x) - \sin x \cos x$$

$$= -\cos x \ln(\sec x + \tan x)$$

and $y = y_p + y_g$

$$\boxed{y = c_1 \cos x + c_2 \sin x - \cos x \ln(\sec x + \tan x)}$$

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